Simplification of Boolean Functions

3-1 THE MAP METHOD

The complexity of the digital logic gates that implement a Boolean function is directly related to the complexity of the algebraic expression from which the function is implemented. Although the truth table representation of a function is unique, expressed algebraically, it can appear in many different forms. Boolean functions may be simplified by algebraic means as discussed in Section 2-4. However, this procedure of minimization is awkward because it lacks specific rules to predict each succeeding step in the manipulative process. The map method provides a simple straightforward procedure for minimizing Boolean functions. This method may be regarded either as a pictorial form of a truth table or as an extension of the Venn diagram. The map method, first proposed by Veitch and modified by Karnaugh, is also known as the "Veitch diagram" or the "Karnaugh map."

The map is a diagram made up of squares. Each square represents one minterm. Since any Boolean function can be expressed as a sum of minterms, it follows that a Boolean function is recognized graphically in the map from the area enclosed by those squares whose minterms are included in the function. In fact, the map presents a visual diagram of all possible ways a function may be expressed in a standard form. By recognizing various patterns, the user can derive alternative algebraic expressions for the same function, from which he can select the simplest one. We shall assume that the simplest algebraic expression is any one in a sum of products or product of sums that has a minimum number of literals. (This expression is not necessarily unique.)

3-2 TWO- AND THREE-VARIABLE MAPS

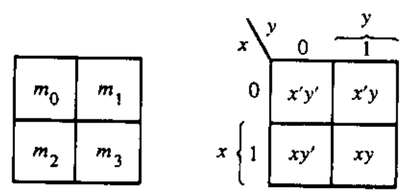
A two-variable map is shown in Fig. 3-1(a). There are four minterms for two variables; hence, the map consists of four squares, one for each minterm. The map is redrawn in (b) to show the relationship between the squares and the two variables. The 0's and 1's marked for each row and each column designate the values of variables x and y, respectively. Notice that x appears primed in row 0 and unprimed in row 1. Similarly, y appears primed in column 0 and unprimed in column 1.

If we mark the squares whose minterms belong to a given function, the two-variable map becomes another useful way to represent any one of the 16 Boolean functions of two variables. As an example, the function xy is shown in Fig. 3-2(a). Since xy is equal to m_3, a 1 is placed inside the square that belongs to m_3. Similarly, the function $x + y$ is represented in the map of Fig. 3-2(b) by three squares marked with 1's. These squares are found from the minterms of the function:

$$x + y = x'y + xy' + xy = m_1 + m_2 + m_3$$

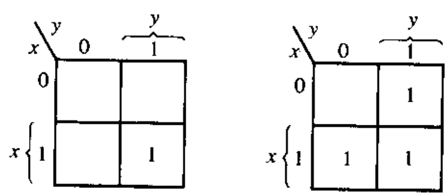
The three squares could have also been determined from the intersection of variable x in the second row and variable y in the second column, which encloses the area belonging to x or y.

A three-variable map is shown in Fig. 3-3. There are eight minterms for three binary variables. Therefore, a map consists of eight squares. Note that the minterms are not arranged in a binary sequence, but in a sequence similar to the Gray code listed in Table 1-4. The characteristic of this sequence is that only one bit changes from 1 to 0



(a) (b)

FIGURE 3-1
Two-variable map



(a) xy (b) $x + y$

FIGURE 3-2
Representation of functions in the map

m_0	m_1	m_3	m_2
m_4	m_5	m_7	m_6

(a)

$x \backslash yz$	00	01	11 (y)	10 (y)
0	$x'y'z'$	$x'y'z$	$x'yz$	$x'yz'$
x 1	$xy'z'$	$xy'z$	xyz	xyz'
		z	z	

(b)

FIGURE 3-3
Three-variable map

or from 0 to 1 in the listing sequence. The map drawn in part (b) is marked with numbers in each row and each column to show the relationship between the squares and the three variables. For example, the square assigned to m_5 corresponds to row 1 and column 01. When these two numbers are concatenated, they give the binary number 101, whose decimal equivalent is 5. Another way of looking at square $m_5 = xy'z$ is to consider it to be in the row marked x and the column belonging to $y'z$ (column 01). Note that there are four squares where each variable is equal to 1 and four where each is equal to 0. The variable appears unprimed in those four squares where it is equal to 1 and primed in those squares where it is equal to 0. For convenience, we write the variable with its letter symbol under the four squares where it is unprimed.

To understand the usefulness of the map for simplifying Boolean functions, we must recognize the basic property possessed by adjacent squares. Any two adjacent squares in the map differ by only one variable, which is primed in one square and unprimed in the other. For example, m_5 and m_7 lie in two adjacent squares. Variable y is primed in m_5 and unprimed in m_7, whereas the other two variables are the same in both squares. From the postulates of Boolean algebra, it follows that the sum of two minterms in adjacent squares can be simplified to a single AND term consisting of only two literals. To clarify this, consider the sum of two adjacent squares such as m_5 and m_7:

$$m_5 + m_7 = xy'z + xyz = xz(y' + y) = xz$$

Here the two squares differ by the variable y, which can be removed when the sum of the two minterms is formed. Thus, any two minterms in adjacent squares that are ORed together will cause a removal of the different variable. The following example explains the procedure for minimizing a Boolean function with a map.

Example 3-1 Simplify the Boolean function

$$F(x, y, z) = \Sigma(2, 3, 4, 5)$$

First, a 1 is marked in each minterm that represents the function. This is shown in Fig. 3-4, where the squares for minterms 010, 011, 100, and 101 are marked with 1's. The next step is to find possible adjacent squares. These are indicated in the map by two rectangles, each enclosing two 1's. The upper right rectangle represents the area en-

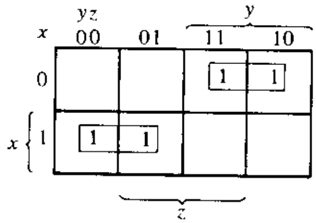


FIGURE 3-4
Map for Example 3-1; $F(x, y, z) = \Sigma(2, 3, 4, 5) = x'y + xy'$

closed by $x'y$. This is determined by observing that the two-square area is in row 0, corresponding to x', and the last two columns, corresponding to y. Similarly, the lower left rectangle represents the product term xy'. (The second row represents x and the two left columns represent y'.) The logical sum of these two product terms gives the simplified expression

$$F = x'y + xy'$$ ■

There are cases where two squares in the map are considered to be adjacent even though they do not touch each other. In Fig. 3-3, m_0 is adjacent to m_2 and m_4 is adjacent to m_6 because the minterms differ by one variable. This can be readily verified algebraically.

$$m_0 + m_2 = x'y'z' + x'yz' = x'z'(y' + y) = x'z'$$

$$m_4 + m_6 = xy'z' + xyz' = xz' + (y' + y) = xz'$$

Consequently, we must modify the definition of adjacent squares to include this and other similar cases. This is done by considering the map as being drawn on a surface where the right and left edges touch each other to form adjacent squares.

Example 3-2

Simplify the Boolean function

$$F(x, y, z) = \Sigma(3, 4, 6, 7)$$

The map for this function is shown in Fig. 3-5. There are four squares marked with 1's, one for each minterm of the function. Two adjacent squares are combined in the third column to give a two-literal term yz. The remaining two squares with 1's are also adjacent by the new definition and are shown in the diagram with their values enclosed in half rectangles. These two squares when combined, give the two-literal term xz'. The simplified function becomes

$$F = yz + xz'$$ ■

Consider now any combination of four adjacent squares in the three-variable map. Any such combination represents the logical sum of four minterms and results in an ex-

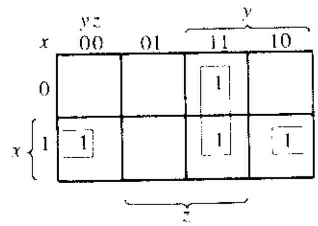


FIGURE 3-5
Map for Example 3-2; $F(x, y, z) = \Sigma(3, 4, 6, 7) = yz + xz'$

pression of only one literal. As an example, the logical sum of the four adjacent minterms 0, 2, 4, and 6 reduces to a single literal term z'.

$$\begin{aligned} m_0 + m_2 + m_4 + m_6 &= x'y'z' + x'yz' + xy'z' + xyz' \\ &= x'z'(y' + y) + xz'(y' + y) \\ &= x'z' + xz' = z'(x' + x) = z' \end{aligned}$$

The number of adjacent squares that may be combined must always represent a number that is a power of two such as 1, 2, 4, and 8. As a larger number of adjacent squares are combined, we obtain a product term with fewer literals.

One square represents one minterm, giving a term of three literals.

Two adjacent squares represent a term of two literals.

Four adjacent squares represent a term of one literal.

Eight adjacent squares encompass the entire map and produce a function that is always equal to 1.

Example 3-3

Simplify the Boolean function

$$F(x, y, z) = \Sigma(0, 2, 4, 5, 6)$$

The map for F is shown in Fig. 3-6. First, we combine the four adjacent squares in the first and last columns to give the single literal term z'. The remaining single square representing minterm 5 is combined with an adjacent square that has already been used once. This is not only permissible, but rather desirable since the two adjacent squares give the two-literal term xy' and the single square represents the three-literal minterm $xy'z$. The simplified function is

$$F = z' + xy'$$ ■

If a function is not expressed in sum of minterms, it is possible to use the map to obtain the minterms of the function and then simplify the function to an expression with a minimum number of terms. It is necessary to make sure that the algebraic ex-

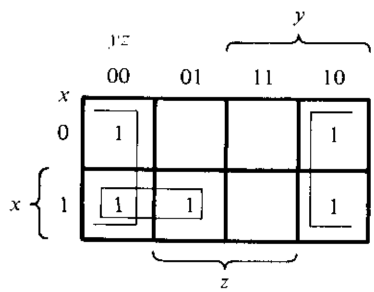


FIGURE 3-6
Map for Example 3-3; $F(x, y, z) = \Sigma\,(0, 2, 4, 5, 6) = z' + xy'$

pression is in sum of products form. Each product term can be plotted in the map in one, two, or more squares. The minterms of the function are then read directly from the map.

Example 3-4

Given the following Boolean function:

$$F = A'C + A'B + AB'C + BC$$

(a) Express it in sum of minterms.
(b) Find the minimal sum of products expression.

Three product terms in the expression have two literals and are represented in a three-variable map by two squares each. The two squares corresponding to the first term $A'C$ are found in Fig. 3-7 from the coincidence of A' (first row) and C (two middle columns) to give squares 001 and 011. Note that when marking 1's in the squares, it is possible to find a 1 already placed there from a preceding term. This happens with the second term $A'B$, which has 1's in squares 011 and 010, but square 011 is common with the first term $A'C$, so only one 1 is marked in it. Continuing in this fashion, we determine that the term $AB'C$ belongs in square 101, corresponding to minterm 5, and the term BC has two 1's in squares 011 and 111. The function has a total of five minterms, as indicated by the five 1's in the map of Fig. 3-7. The minterms are read

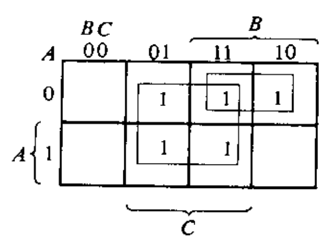


FIGURE 3-7
Map for Example 3-4; $A'C + A'B + AB'C + BC = C + A'B$

directly from the map to be 1, 2, 3, 5, and 7. The function can be expressed in sum of minterms form:

$$F(A, B, C) = \Sigma(1, 2, 3, 5, 7)$$

The sum of products expression as originally given has too many terms. It can be simplified, as shown in the map, to an expression with only two terms:

$$F = C + A'B$$

■

3-3 FOUR-VARIABLE MAP

The map for Boolean functions of four binary variables is shown in Fig. 3-8. In (a) are listed the 16 minterms and the squares assigned to each. In (b) the map is redrawn to show the relationship with the four variables. The rows and columns are numbered in a reflected-code sequence, with only one digit changing value between two adjacent rows or columns. The minterm corresponding to each square can be obtained from the concatenation of the row number with the column number. For example, the numbers of the third row (11) and the second column (01), when concatenated, give the binary number 1101, the binary equivalent of decimal 13. Thus, the square in the third row and second column represents minterm m_{13}.

The map minimization of four-variable Boolean functions is similar to the method used to minimize three-variable functions. Adjacent squares are defined to be squares next to each other. In addition, the map is considered to lie on a surface with the top and bottom edges, as well as the right and left edges, touching each other to form adjacent squares. For example, m_0 and m_2 form adjacent squares, as do m_3 and m_{11}. The combination of adjacent squares that is useful during the simplification process is easily determined from inspection of the four-variable map:

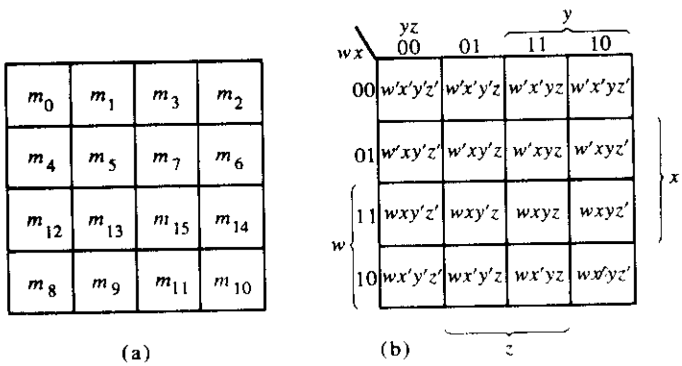


FIGURE 3-8
Four-variable map

One square represents one minterm, giving a term of four literals.
Two adjacent squares represent a term of three literals.
Four adjacent squares represent a term of two literals.
Eight adjacent squares represent a term of one literal.
Sixteen adjacent squares represent the function equal to 1.

No other combination of squares can simplify the function. The following two examples show the procedure used to simplify four-variable Boolean functions.

Example 3-5

Simplify the Boolean function

$$F(w, x, y, z) = \Sigma(0, 1, 2, 4, 5, 6, 8, 9, 12, 13, 14)$$

Since the function has four variables, a four-variable map must be used. The minterms listed in the sum are marked by 1's in the map of Fig. 3-9. Eight adjacent squares marked with 1's can be combined to form the one literal term y'. The remaining three 1's on the right cannot be combined to give a simplified term. They must be combined as two or four adjacent squares. The larger the number of squares combined, the smaller the number of literals in the term. In this example, the top two 1's on the right are combined with the top two 1's on the left to give the term $w'z'$. Note that it is permissible to use the same square more than once. We are now left with a square marked by 1 in the third row and fourth column (square 1110). Instead of taking this square alone (which will give a term of four literals), we combine it with squares already used to form an area of four adjacent squares. These squares comprise the two middle rows and the two end columns, giving the term xz'. The simplified function is

$$F = y' + w'z' + xz'$$ ■

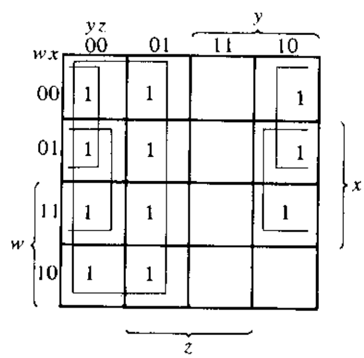


FIGURE 3-9
Map for Example 3-5; $F(w, x, y, z) = \Sigma(0, 1, 2, 4, 5, 6, 8, 9, 12, 13, 14) = y' + w'z' + xz'$

Example 3-6

Simplify the Boolean function

$$F = A'B'C' + B'CD' + A'BCD' + AB'C'$$

The area in the map covered by this function consists of the squares marked with 1's in Fig. 3-10. This function has four variables and, as expressed, consists of three terms, each with three literals, and one term of four literals. Each term of three literals is represented in the map by two squares. For example, $A'B'C'$ is represented in squares 0000 and 0001. The function can be simplified in the map by taking the 1's in the four corners to give the term $B'D'$. This is possible because these four squares are adjacent when the map is drawn in a surface with top and bottom or left and right edges touching one another. The two left-hand 1's in the top row are combined with the two 1's in the bottom row to give the term $B'C'$. The remaining 1 may be combined in a two-square area to give the term $A'CD'$. The simplified function is

$$F = B'D' + B'C' + A'CD'$$

■

Prime Implicants

When choosing adjacent squares in a map, we must ensure that all the minterms of the function are covered when combining the squares. At the same time, it is necessary to minimize the number of terms in the expression and avoid any redundant terms whose minterms are already covered by other terms. Sometimes there may be two or more expressions that satisfy the simplification criteria. The procedure for combining squares in the map may be made more systematic if we understand the meaning of the terms referred to as prime implicant and essential prime implicant. A *prime implicant* is a product term obtained by combining the maximum possible number of adjacent squares in the map. If a minterm in a square is covered by only one prime implicant, that prime implicant is said to be *essential*. A more satisfactory definition of prime implicant is given in Section 3-10. Here we will use it to help us find all possible simplified expressions of a Boolean function by means of a map.

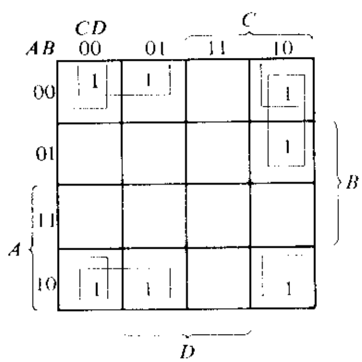


FIGURE 3-10
Map for Example 3-6: $A'B'C' + B'CD' + A'BCD' + AB'C' = B'D' + B'C' + A'CD'$

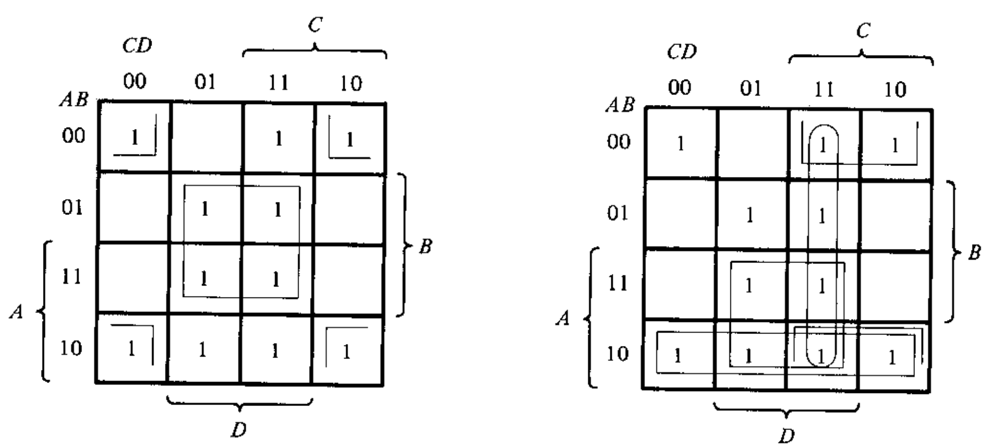
The prime implicants of a function can be obtained from the map by combining all possible maximum numbers of squares. This means that a single 1 on a map represents a prime implicant if it is not adjacent to any other 1's. Two adjacent 1's form a prime implicant provided they are not within a group of four adjacent squares. Four adjacent 1's form a prime implicant if they are not within a group of eight adjacent squares, and so on. The essential prime implicants are found by looking at each square marked with a 1 and checking the number of prime implicants that cover it. The prime implicant is essential if it is the only prime implicant that covers the minterm.

Consider the following four-variable Boolean function:

$$F(A, B, C, D) = \Sigma(0, 2, 3, 5, 7, 8, 9, 10, 11, 13, 15)$$

The minterms of the function are marked with 1's in the maps of Fig. 3-11. Part (a) of the figure shows two essential prime implicants. One term is essential because there is only one way to include minterms m_0 within four adjacent squares. These four squares define the term $B'D'$. Similarly, there is only one way that minterm m_5 can be combined with four adjacent squares and this gives the second term BD. The two essential prime implicants cover eight minterms. The remaining three minterms, m_3, m_9, and m_{11}, must be considered next.

Figure 3-11(b) shows all possible ways that the three minterms can be covered with prime implicants. Minterm m_3 can be covered with either prime implicant CD or $B'C$. Minterm m_9 can be covered with either AD or AB'. Minterm m_{11} is covered with any one of the four prime implicants. The simplified expression is obtained from the logical sum of the two essential prime implicants and any two prime implicants that cover minterms m_3, m_9, and m_{11}. There are four possible ways that the function can be expressed with four product terms of two literals each:



(a) Essential prime implicants BD and $B'D'$

(b) Prime implicants CD, $B'C$, AD, and AB'

FIGURE 3-11

Simplification using prime implicants

$$\begin{aligned} F &= BD + B'D' + CD + AD \\ &= BD + B'D' + CD + AB' \\ &= BD + B'D' + B'C + AD \\ &= BD + B'D' + B'C + AB' \end{aligned}$$

The above example has demonstrated that the identification of the prime implicants in the map helps in determining the alternatives that are available for obtaining a simplified expression.

The procedure for finding the simplified expression from the map requires that we first determine all the essential prime implicants. The simplified expression is obtained from the logical sum of all the essential prime implicants plus other prime implicants that may be needed to cover any remaining minterms not covered by the essential prime implicants. Occasionally, there may be more than one way of combining squares and each combination may produce an equally simplified expression.

3-4 FIVE-VARIABLE MAP

Maps for more than four variables are not as simple to use. A five-variable map needs 32 squares and a six-variable map needs 64 squares. When the number of variables becomes large, the number of squares becomes excessively large and the geometry for combining adjacent squares becomes more involved.

The five-variable map is shown in Fig. 3-12. It consists of 2 four-variable maps with variables A, B, C, D, and E. Variable A distinguishes between the two maps, as indicated on the top of the diagram. The left-hand four-variable map represents the 16

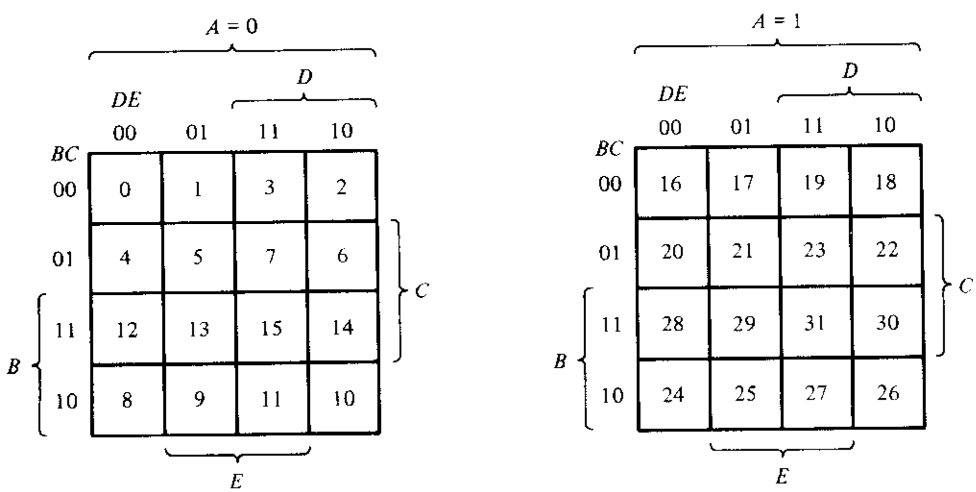


FIGURE 3-12
Five-variable map

squares where $A = 0$, and the other four-variable map represents the squares where $A = 1$. Minterms 0 through 15 belong with $A = 0$ and minterms 16 through 31 with $A = 1$. Each four-variable map retains the previously defined adjacency when taken separately. In addition, each square in the $A = 0$ map is adjacent to the corresponding square in the $A = 1$ map. For example, minterm 4 is adjacent to minterm 20 and minterm 15 to 31. The best way to visualize this new rule for adjacent squares is to consider the two half maps as being one on top of the other. Any two squares that fall one over the other are considered adjacent.

By following the procedure used for the five-variable map, it is possible to construct a six-variable map with 4 four-variable maps to obtain the required 64 squares. Maps with six or more variables need too many squares and are impractical to use. The alternative is to employ computer programs specifically written to facilitate the simplification of Boolean functions with a large number of variables.

From inspection, and taking into account the new definition of adjacent squares, it is possible to show that any 2^k adjacent squares, for $k = 0, 1, 2, \ldots, n$, in an n-variable map, will represent an area that gives a term of $n - k$ literals. For the above statement to have any meaning, n must be larger than k. When $n = k$, the entire area of the map is combined to give the identity function. Table 3-1 shows the relationship between the number of adjacent squares and the number of literals in the term. For example, eight adjacent squares combine an area in the five-variable map to give a term of two literals.

TABLE 3-1
The Relationship Between the Number of Adjacent Squares and the Number of Literals in the Term

	Number of adjacent squares	Number of literals in a term in an n-variable map					
k	2^k	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$
0	1	2	3	4	5	6	7
1	2	1	2	3	4	5	6
2	4	0	1	2	3	4	5
3	8		0	1	2	3	4
4	16			0	1	2	3
5	32				0	1	2
6	64					0	1

Example 3-7 Simplify the Boolean function

$$F(A, B, C, D, E) = (0, 2, 4, 6, 9, 13, 21, 23, 25, 29, 31)$$

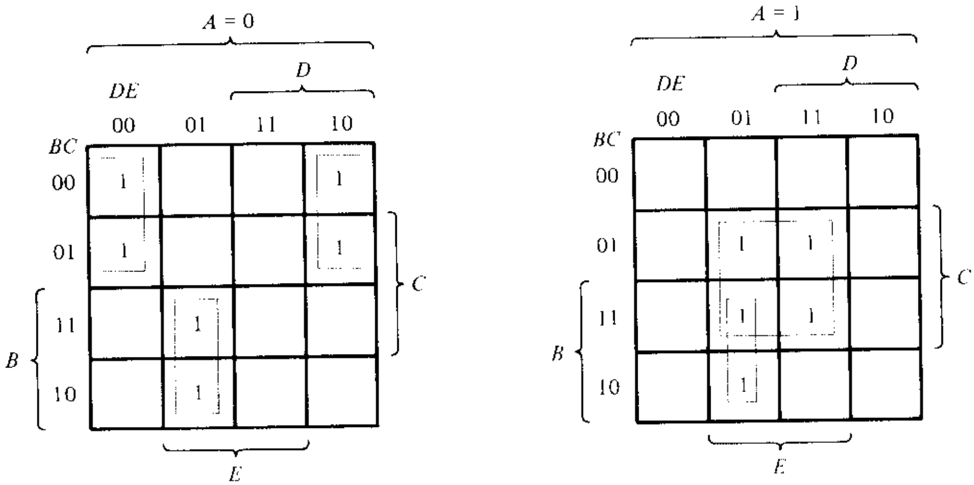


FIGURE 3-13
Map for Example 3-7: $F = A'B'E' + BD'E + ACE$

The five-variable map for this function is shown in Fig. 3-13. There are six minterms from 0 to 15 that belong to the part of the map with $A = 0$. The other five minterms belong with $A = 1$. Four adjacent squares in the $A = 0$ map are combined to give the three-literal term $A'B'E'$. Note that it is necessary to include A' with the term because all the squares are associated with $A = 0$. The two squares in column 01 and the last two rows are common to both parts of the map. Therefore, they constitute four adjacent squares and give the three-literal term $BD'E$. Variable A is not included here because the adjacent squares belong to both $A = 0$ and $A = 1$. The term ACE is obtained from the four adjacent squares that are entirely within the $A = 1$ map. The simplified function is the logical sum of the three terms:

$$F = A'B'E' + BD'E + ACE$$ ■

3-5 PRODUCT OF SUMS SIMPLIFICATION

The minimized Boolean functions derived from the map in all previous examples were expressed in the sum of products form. With a minor modification, the product of sums form can be obtained.

The procedure for obtaining a minimized function in product of sums follows from the basic properties of Boolean functions. The 1's placed in the squares of the map represent the minterms of the function. The minterms not included in the function denote the complement of the function. From this we see that the complement of a function is represented in the map by the squares not marked by 1's. If we mark the empty squares by 0's and combine them into valid adjacent squares, we obtain a simplified expression of the complement of the function, i.e., of F'. The complement of F' gives us

back the function F. Because of the generalized DeMorgan's theorem, the function so obtained is automatically in the product of sums form. The best way to show this is by example.

Example 3-8

Simplify the following Boolean function in (a) sum of products and (b) product of sums.

$$F(A, B, C, D) = \Sigma(0, 1, 2, 5, 8, 9, 10)$$

The 1's marked in the map of Fig. 3-14 represent all the minterms of the function. The squares marked with 0's represent the minterms not included in F and, therefore, denote the complement of F. Combining the squares with 1's gives the simplified function in sum of products:

(a) $$F = B'D' + B'C' + A'C'D$$

If the squares marked with 0's are combined, as shown in the diagram, we obtain the simplified complemented function:

$$F' = AB + CD + BD'$$

Applying DeMorgan's theorem (by taking the dual and complementing each literal as described in Section 2-4), we obtain the simplified function in product of sums:

(b) $$F = (A' + B')(C' + D')(B' + D)$$ ■

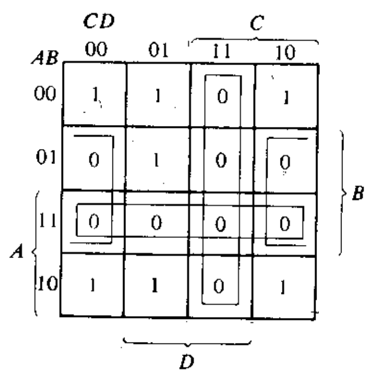
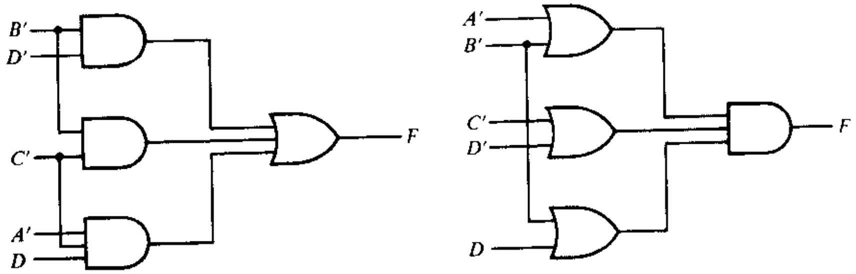


FIGURE 3-14
Map for Example 3-8; $F(A, B, C, D) = \Sigma(0, 1, 2, 5, 8, 9, 10) = B'D' + B'C' + A'C'D = (A' + B')(C' + D')(B' + D)$

The implementation of the simplified expressions obtained in Example 3-8 is shown in Fig. 3-15. The sum of products expression is implemented in (a) with a group of AND gates, one for each AND term. The outputs of the AND gates are connected to the inputs of a single OR gate. The same function is implemented in (b) in its product of sums form with a group of OR gates, one for each OR term. The outputs of the OR



(a) $F = B'D' + B'C' + A'C'D$ (b) $F = (A' - B')(C' + D')(B' + D)$

FIGURE 3-15

Gate implementation of the function of Example 3-8

gates are connected to the inputs of a single AND gate. In each case, it is assumed that the input variables are directly available in their complement, so inverters are not needed. The configuration pattern established in Fig. 3-15 is the general form by which any Boolean function is implemented when expressed in one of the standard forms. AND gates are connected to a single OR gate when in sum of products; OR gates are connected to a single AND gate when in product of sums. Either configuration forms two levels of gates. Thus, the implementation of a function in a standard form is said to be a two-level implementation.

Example 3-8 showed the procedure for obtaining the product of sums simplification when the function is originally expressed in the sum of minterms canonical form. The procedure is also valid when the function is originally expressed in the product of maxterms canonical form. Consider, for example, the truth table that defines the function F in Table 3-2. In sum of minterms, this function is expressed as

$$F(x, y, z) = \Sigma(1, 3, 4, 6)$$

In product of maxterms, it is expressed as

$$F(x, y, z) = \Pi(0, 2, 5, 7)$$

TABLE 3-2
Truth Table of Function F

x	y	z	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

In other words, the 1's of the function represent the minterms, and the 0's represent the maxterms. The map for this function is shown in Fig. 3-16. One can start simplifying this function by first marking the 1's for each minterm that the function is a 1. The remaining squares are marked by 0's. If, on the other hand, the product of maxterms is initially given, one can start marking 0's in those squares listed in the function; the remaining squares are then marked by 1's. Once the 1's and 0's are marked, the function can be simplified in either one of the standard forms. For the sum of products, we combine the 1's to obtain

$$F = x'z + xz'$$

For the product of sums, we combine the 0's to obtain the simplified complemented function:

$$F' = xz + x'z'$$

which shows that the exclusive-OR function is the complement of the equivalence function (Section 2-6). Taking the complement of F', we obtain the simplified function in product of sums:

$$F = (x' + z')(x + z)$$

To enter a function expressed in product of sums in the map, take the complement of the function and from it find the squares to be marked by 0's. For example, the function

$$F = (A' + B' + C')(B + D)$$

can be entered in the map by first taking its complement:

$$F' = ABC + B'D'$$

and then marking 0's in the squares representing the minterms of F'. The remaining squares are marked with 1's.

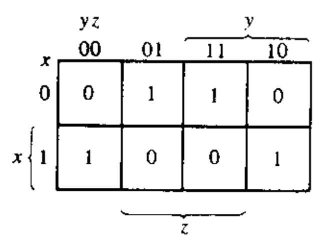


FIGURE 3-16
Map for the function of Table 3-2

3-6 NAND AND NOR IMPLEMENTATION

Digital circuits are more frequently constructed with NAND or NOR gates than with AND and OR gates. NAND and NOR gates are easier to fabricate with electronic components and are the basic gates used in all IC digital logic families. Because of the prominence of NAND and NOR gates in the design of digital circuits, rules and procedures have been developed for the conversion from Boolean functions given in terms of AND, OR, and NOT into equivalent NAND and NOR logic diagrams. The procedure for two-level implementation is presented in this section. Multilevel implementation is discussed in Section 4-7.

To facilitate the conversion to NAND and NOR logic, it is convenient to define two other graphic symbols for these gates. Two equivalent symbols for the NAND gate are shown in Fig. 3-17(a). The AND-invert symbol has been defined previously and consists of an AND graphic symbol followed by a small circle. Instead, it is possible to represent a NAND gate by an OR graphic symbol preceded by small circles in all the inputs. The invert-OR symbol for the NAND gate follows from DeMorgan's theorem and from the convention that small circles denote complementation.

Similarly, there are two graphic symbols for the NOR gate, as shown in Fig. 3-17(b). The OR-invert is the conventional symbol. The invert-AND is a convenient alternative that utilizes DeMorgan's theorem and the convention that small circles in the inputs denote complementation.

A one-input NAND or NOR gate behaves like an inverter. As a consequence, an inverter gate can be drawn in three different ways, as shown in Fig. 3-17(c). The small circles in all inverter symbols can be transferred to the input terminal without changing the logic of the gate.

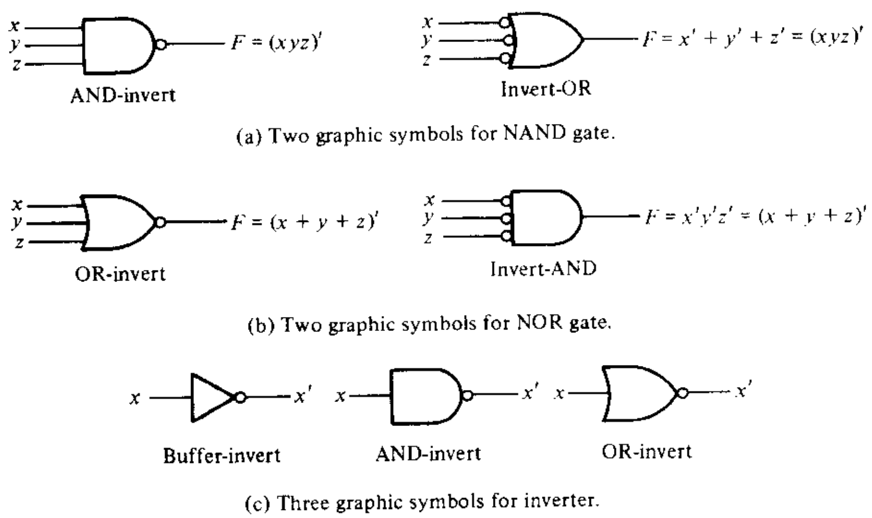


FIGURE 3-17
Graphic symbols for NAND and NOR gates

It should be pointed out that the alternate symbols for the NAND and NOR gates could be drawn with small triangles in all input terminals instead of the circles. A small triangle is a negative-logic polarity indicator (see Section 2-8 and Fig. 2-11). With small triangles in the input terminals, the graphic symbol denotes a negative-logic polarity for the inputs, but the output of the gate (not having a triangle) would have a positive-logic assignment. In this book, we prefer to stay with positive logic throughout and employ small circles when necessary to denote complementation.

NAND Implementation

The implementation of a Boolean function with NAND gates requires that the function be simplified in the sum of products form. To see the relationship between a sum of products expression and its equivalent NAND implementation, consider the logic diagrams of Fig. 3-18. All three diagrams are equivalent and implement the function:

$$F = AB + CD + E$$

The function is implemented in Fig. 3-18(a) in sum of products form with AND and OR gates. In (b) the AND gates are replaced by NAND gates and the OR gate is replaced by a NAND gate with an invert-OR symbol. The single variable E is complemented and applied to the second-level invert-OR gate. Remember that a small circle denotes complementation. Therefore, two circles on the same line represent double complementation and both can be removed. The complement of E goes through a small

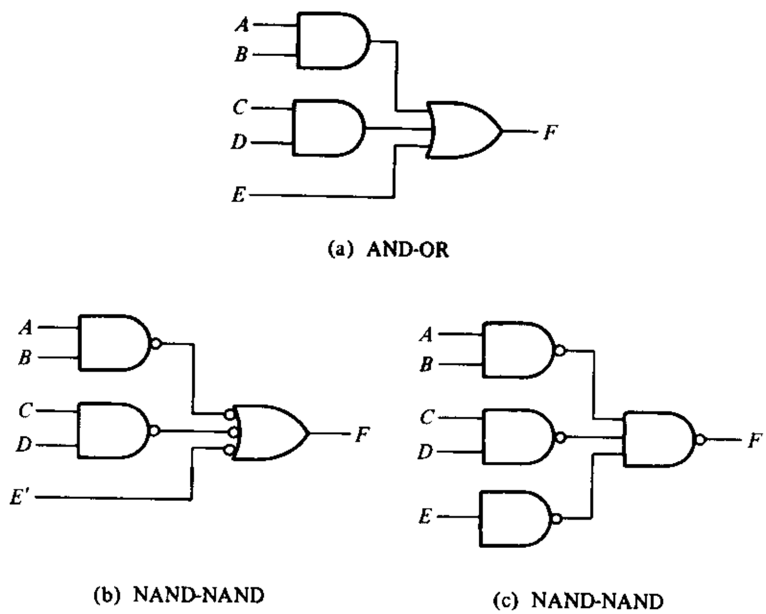


FIGURE 3-18
Three ways to implement $F = AB + CD + E$

circle that complements the variable again to produce the normal value of E. Removing the small circles in the gates of Fig. 3-18(b) produces the circuit in (a). Therefore, the two diagrams implement the same function and are equivalent.

In Fig. 3-18(c), the output NAND gate is redrawn with the conventional symbol. The one-input NAND gate complements variable E. It is possible to remove this inverter and apply E' directly to the input of the second-level NAND gate. The diagram in (c) is equivalent to the one in (b), which in turn is equivalent to the diagram in (a). Note the similarity between the diagrams in (a) and (c). The AND and OR gates have been changed to NAND gates, but an additional NAND gate has been included with the single variable E. When drawing NAND logic diagrams, the circuit shown in either (b) or (c) is acceptable. The one in (b), however, represents a more direct relationship to the Boolean expression it implements.

The NAND implementation in Fig. 3-18(c) can be verified algebraically. The NAND function it implements can be easily converted to a sum of products form by using DeMorgan's theorem:

$$F = [(AB)' \cdot (CD)' \cdot E']' = AB + CD + E$$

From the transformation shown in Fig. 3-18, we conclude that a Boolean function can be implemented with two levels of NAND gates. The rule for obtaining the NAND logic diagram from a Boolean function is as follows:

1. Simplify the function and express it in sum of products.
2. Draw a NAND gate for each product term of the function that has at least two literals. The inputs to each NAND gate are the literals of the term. This constitutes a group of first-level gates.
3. Draw a single NAND gate (using the AND-invert or invert-OR graphic symbol) in the second level, with inputs coming from outputs of first-level gates.
4. A term with a single literal requires an inverter in the first level or may be complemented and applied as an input to the second-level NAND gate.

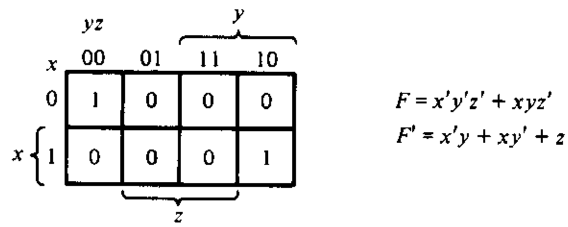
Before applying these rules to a specific example, it should be mentioned that there is a second way to implement a Boolean function with NAND gates. Remember that if we combine the 0's in a map, we obtain the simplified expression of the *complement* of the function in sum of products. The complement of the function can then be implemented with two levels of NAND gates using the rules stated above. If the normal output is desired, it would be necessary to insert a one-input NAND or inverter gate to generate the true value of the output variable. There are occasions where the designer may want to generate the complement of the function; so this second method may be preferable.

Example 3-9

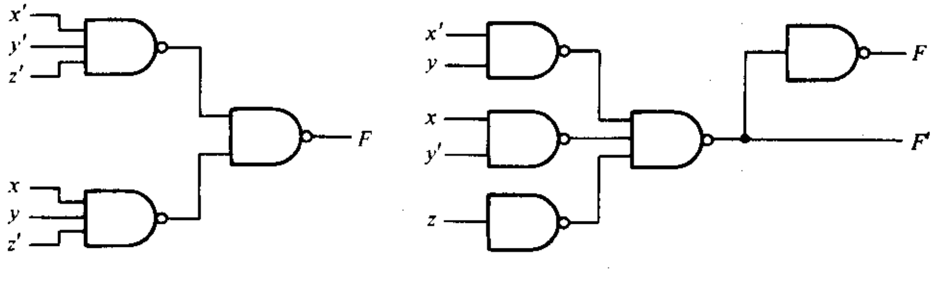
Implement the following function with NAND gates:

$$F(x, y, z) = \Sigma(0, 6)$$

The first step is to simplify the function in sum of products form. This is attempted with the map shown in Fig. 3-19(a). There are only two 1's in the map, and they can-



(a) Map simplification in sum of products.



(b) $F = x'y'z' + xyz'$

(c) $F' = x'y + xy' + z$

FIGURE 3-19
Implementation of the function of Example 3-9 with NAND gates

not be combined. The simplified function in sum of products for this example is

$$F = x'y'z' + xyz'$$

The two-level NAND implementation is shown in Fig. 3-19(b). Next we try to simplify the complement of the function in sum of products. This is done by combining the 0's in the map:

$$F' = x'y + xy' + z$$

The two-level NAND gate for generating F' is shown in Fig. 3-19(c). If output F is required, it is necessary to add a one-input NAND gate to invert the function. This gives a three-level implementation. In each case, it is assumed that the input variables are available in both the normal and complement forms. If they were available in only one form, it would be necessary to insert inverters in the inputs, which would add another level to the circuits. The one-input NAND gate associated with the single variable z can be removed provided the input is changed to z'. ■

NOR Implementation

The NOR function is the dual of the NAND function. For this reason, all procedures and rules for NOR logic are the duals of the corresponding procedures and rules developed for NAND logic.

The implementation of a Boolean function with NOR gates requires that the function be simplified in product of sums form. A product of sums expression specifies a group of OR gates for the sum terms, followed by an AND gate to produce the product. The transformation from the OR-AND to the NOR-NOR diagram is depicted in Fig. 3-20. It is similar to the NAND transformation discussed previously, except that now we use the product of sums expression

$$F = (A + B)(C + D)E$$

The rule for obtaining the NOR logic diagram from a Boolean function can be derived from this transformation. It is similar to the three-step NAND rule, except that the simplified expression must be in the product of sums and the terms for the first-level NOR gates are the sum terms. A term with a single literal requires a one-input NOR or inverter gate or may be complemented and applied directly to the second-level NOR gate.

A second way to implement a function with NOR gates would be to use the expression for the complement of the function in product of sums. This will give a two-level implementation for F' and a three-level implementation if the normal output F is required.

To obtain the simplified product of sums from a map, it is necessary to combine the 0's in the map and then complement the function. To obtain the simplified product of sums expression for the complement of the function, it is necessary to combine the 1's in the map and then complement the function. The following example demonstrates the procedure for NOR implementation.

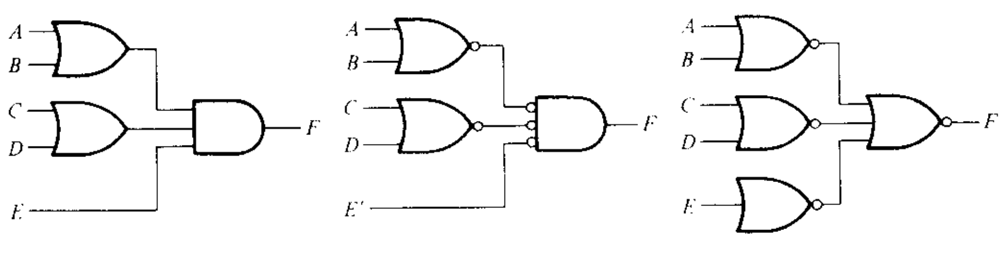
Example 3-10

Implement the function of Example 3-9 with NOR gates.

The map of this function is drawn in Fig. 3-19(a). First, combine the 0's in the map to obtain

$$F' = x'y + xy' + z$$

This is the complement of the function in sum of products. Complement F' to obtain the simplified function in product of sums as required for NOR implementation:



(a) OR-AND (b) NOR-NOR (c) NOR-NOR

FIGURE 3-20

Three ways to implement $F = (A + B)(C + D)E$

$$F = (x + y')(x' + y)z'$$

The two-level implementation with NOR gates is shown in Fig. 3-21(a). The term with a single literal z' requires a one-input NOR or inverter gate. This gate can be removed and input z applied directly to the input of the second-level NOR gate.

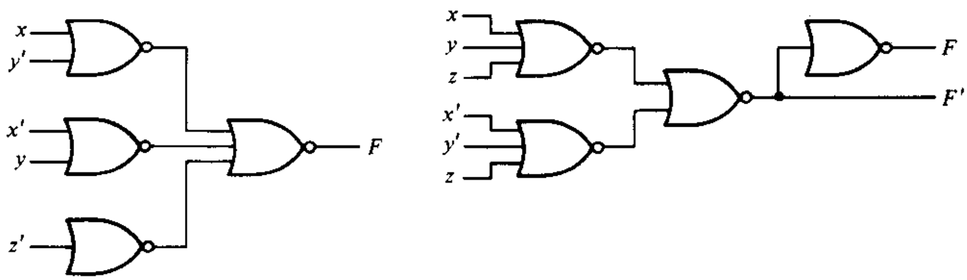
A second implementation is possible from the complement of the function in product of sums. For this case, first combine the 1's in the map to obtain

$$F = x'y'z' + xyz'$$

This is the simplified expression in sum of products. Complement this function to obtain the complement of the function in product of sums as required for NOR implementation:

$$F' = (x + y + z)(x' + y' + z)$$

The two-level implementation for F' is shown in Fig. 3-21(b). If output F is desired, it can be generated with an inverter in the third level. ■



(a) $F = (x + y')(x' + y)z'$ (b) $F' = (x + y + z)(x' + y' + z)$

FIGURE 3-21
Implementation with NOR gates

Table 3-3 summarizes the procedures for NAND or NOR implementation. One should not forget to always simplify the function in order to reduce the number of gates in the implementation. The standard forms obtained from the map-simplification procedures apply directly and are very useful when dealing with NAND or NOR logic.

TABLE 3-3
Rules for NAND and NOR Implementation

Case	Function to simplify	Standard form to use	How to derive	Implement with	Number of levels to F
(a)	F	Sum of products	Combine 1's in map	NAND	2
(b)	F'	Sum of products	Combine 0's in map	NAND	3
(c)	F	Product of sums	Complement F' in (b)	NOR	2
(d)	F'	Product of sums	Complement F in (a)	NOR	3

3-7 OTHER TWO-LEVEL IMPLEMENTATIONS

The types of gates most often found in integrated circuits are NAND and NOR. For this reason, NAND and NOR logic implementations are the most important from a practical point of view. Some NAND or NOR gates (but not all) allow the possibility of a wire connection between the outputs of two gates to provide a specific logic function. This type of logic is called *wired logic*. For example, open-collector TTL NAND gates, when tied together, perform the wired-AND logic. (The open-collector TTL gate is shown in Chapter 10, Fig. 10-11.) The wired-AND logic performed with two NAND gates is depicted in Fig. 3-22(a). The AND gate is drawn with the lines going through the center of the gate to distinguish it from a conventional gate. The wired-AND gate is not a physical gate, but only a symbol to designate the function obtained from the indicated wired connection. The logic function implemented by the circuit of Fig. 3-22(a) is

$$F = (AB)' \cdot (CD)' = (AB + CD)'$$

and is called an AND-OR-INVERT function.

Similarly, the NOR output of ECL gates can be tied together to perform a wired-OR function. The logic function implemented by the circuit of Fig. 3-22(b) is

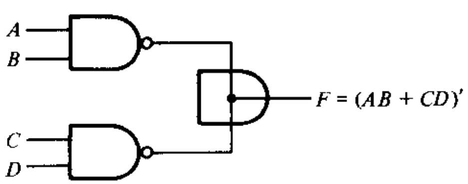
$$F = (A + B)' + (C + D)' = [(A + B)(C + D)]'$$

and is called an OR-AND-INVERT function.

A wired-logic gates does not produce a physical second-level gate since it is just a wire connection. Nevertheless, for discussion purposes, we will consider the circuits of Fig. 3-22 as two-level implementations. The first level consists of NAND (or NOR) gates and the second level has a single AND (or OR) gate. The wired connection in the graphic symbol will be omitted in subsequent discussions.

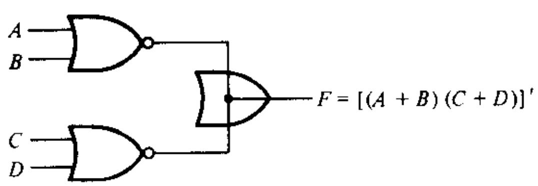
Nondegenerate Forms

It will be instructive from a theoretical point of view to find out how many two-level combinations of gates are possible. We consider four types of gates: AND, OR, NAND, and NOR. If we assign one type of gate for the first level and one type for the second



(a) Wired-AND in open-collector TTL NAND gates.

(AND-OR-INVERT)



(b) Wired-OR in ECL gates

(OR-AND-INVERT)

FIGURE 3-22
Wired logic

level, we find that there are 16 possible combinations of two-level forms. (The same type of gate can be in the first and second levels, as in NAND-NAND implementation.) Eight of these combinations are said to be *degenerate* forms because they degenerate to a single operation. This can be seen from a circuit with AND gates in the first level and an AND gate in the second level. The output of the circuit is merely the AND function of all input variables. The other eight *nondegenerate* forms produce an implementation in sum of products or product of sums. The eight nondegenerate forms are

AND-OR	OR-AND
NAND-NAND	NOR-NOR
NOR-OR	NAND-AND
OR-NAND	AND-NOR

The first gate listed in each of the forms constitutes a first level in the implementation. The second gate listed is a single gate placed in the second level. Note that any two forms listed in the same line are the duals of each other.

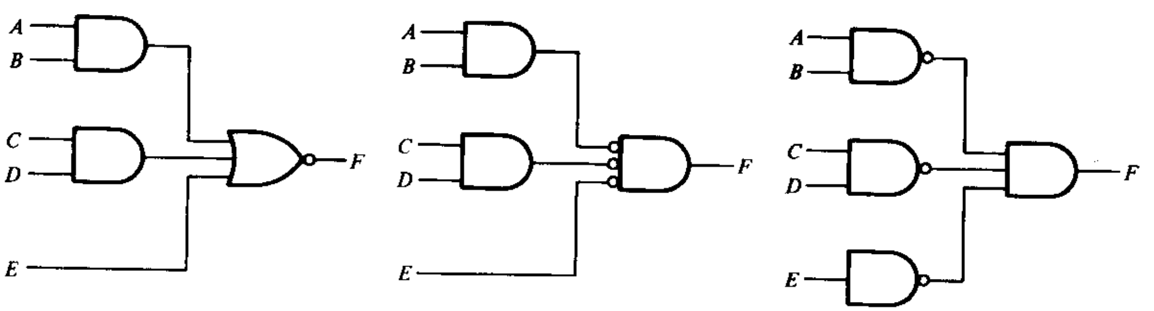
The AND-OR and OR-AND forms are the basic two-level forms discussed in Section 3-5. The NAND-NAND and NOR-NOR were introduced in Section 3-6. The remaining four forms are investigated in this section.

AND-OR-INVERT Implementation

The two forms NAND-AND and AND-NOR are equivalent forms and can be treated together. Both perform the AND-OR-INVERT function, as shown in Fig. 3-23. The AND-NOR form resembles the AND-OR form with an inversion done by the small circle in the output of the NOR gate. It implements the function

$$F = (AB + CD + E)'$$

By using the alternate graphic symbol for the NOR gate, we obtain the diagram of Fig. 3-23(b). Note that the single variable E is *not* complemented because the only change made is in the graphic symbol of the NOR gate. Now we move the circles from



(a) AND-NOR (b) AND-NOR. (c) NAND-AND

FIGURE 3-23

AND-OR-INVERT circuits; $F = (AB + CD + E)'$

the input terminal of the second-level gate to the output terminals of the first-level gates. An inverter is needed for the single variable to maintain the circle. Alternatively, the inverter can be removed provided input E is complemented. The circuit of Fig. 3-23(c) is a NAND-AND form and was shown in Fig. 3-22 to implement the AND-OR-INVERT function.

An AND-OR implementation requires an expression in sum of products. The AND-OR-INVERT implementation is similar except for the inversion. Therefore, if the *complement* of the function is simplified in sum of products (by combining the 0's in the map), it will be possible to implement F' with the AND-OR part of the function. When F' passes through the always present output inversion (the INVERT part), it will generate the ouput F of the function. An example for the AND-OR-INVERT implementation will be shown subsequently.

OR-AND-INVERT Implementation

The OR-NAND and NOR-OR forms perform the OR-AND-INVERT function. This is shown in Fig. 3-24. The OR-NAND form resembles the OR-AND form, except for the inversion done by the circle in the NAND gate. It implements the function

$$F = [(A + B)(C + D)E]'$$

By using the alternate graphic symbol for the NAND gate, we obtain the diagram of Fig. 3-24(b). The circuit in (c) is obtained by moving the small circles from the inputs of the second-level gate to the outputs of the first-level gates. The circuit of Fig. 3-24(c) is a NOR-OR form and was shown in Fig. 3-22 to implement the OR-AND-INVERT function.

The OR-AND-INVERT implementation requires an expression in product of sums. If the complement of the function is simplified in product of sums, we can implement F' with the OR-AND part of the function. When F' passes through the INVERT part, we obtain the complement of F', or F, in the output.

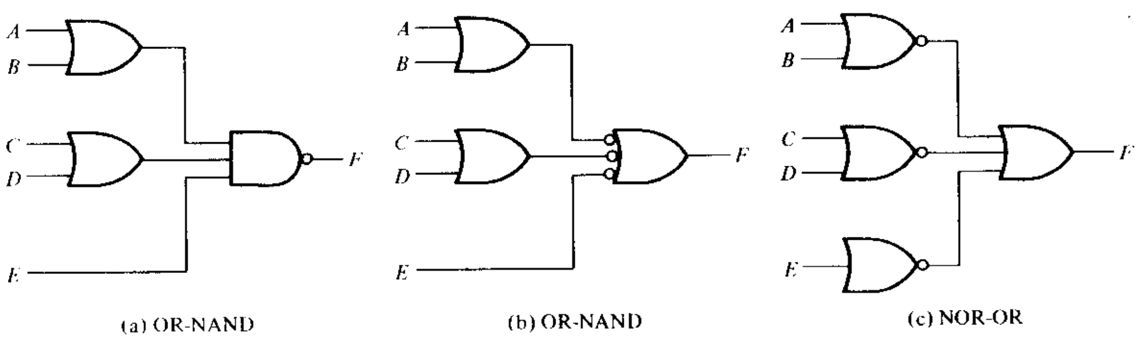


FIGURE 3-24
OR-AND-INVERT circuits; $F = [(A + B)(C + D)E]'$

Tabular Summary and Example

Table 3-4 summarizes the procedures for implementing a Boolean function in any one of the four two-level forms. Because of the INVERT part in each case, it is convenient to use the simplification of F' (the complement) of the function. When F' is implemented in one of these forms, we obtain the complement of the function in the AND-OR or OR-AND form. The four two-level forms invert this function, giving an output that is the complement of F'. This is the normal output F.

TABLE 3-4
Implementation with Other Two-Level Forms

Equivalent nondegenerate form (a)	Equivalent nondegenerate form (b)*	Implements the function	Simplify F' in	To get an output of
AND-NOR	NAND-AND	AND-OR-INVERT	Sum of products by combining 0's in the map	F
OR-NAND	NOR-OR	OR-AND-INVERT	Product of sums by combining 1's in the map and then complementing	F

*Form (b) requires a one-input NAND or NOR (inverter) gate for a single literal term.

Example 3-11 Implement the function of Fig. 3-19(a) with the four two-level forms listed in Table 3-4. The complement of the function is simplified in sum of products by combining the 0's in the map:

$$F' = x'y + xy' + z$$

The normal output for this function can be expressed as

$$F = (x'y + xy' + z)'$$

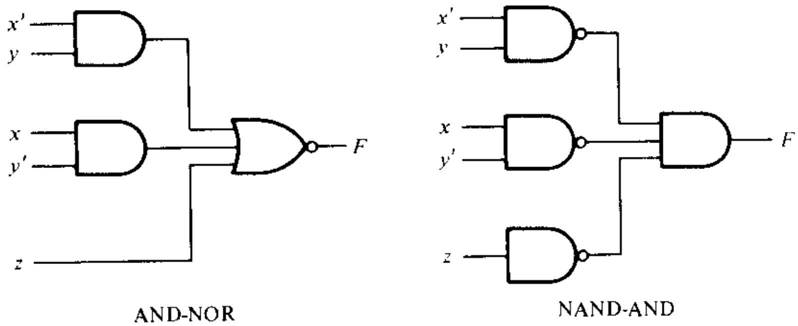
which is in the AND-OR-INVERT form. The AND-NOR and NAND-AND implementations are shown in Fig. 3-25(a). Note that a one-input NAND or inverter gate is needed in the NAND-AND implementation, but not in the AND-NOR case. The inverter can be removed if we apply the input variable z' instead of z.

The OR-AND-INVERT forms require a simplified expression of the complement of the function in product of sums. To obtain this expression, we must first combine the 1's in the map

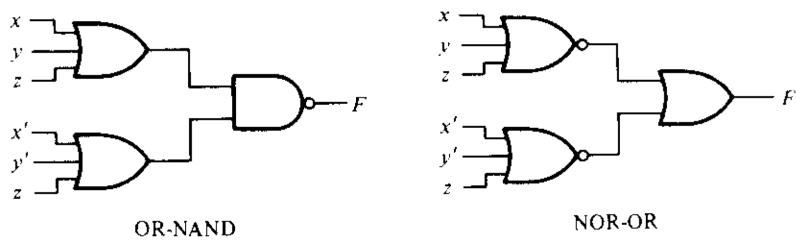
$$F = x'y'z' + xyz'$$

Then we take the complement of the function

$$F' = (x + y + z)(x' + y' + z)$$



(a) $F = (x'y + xy' + z)'$



(b) $F = [(x + y + z)(x' + y' + z)]'$

FIGURE 3-25
Other two-level implementations

The normal output F can now be expressed in the form

$$F = [(x + y + z)(x' + y' + z)]'$$

which is in the OR-AND-INVERT form. From this expression, we can implement the function in the OR-NAND and NOR-OR forms, as shown in Fig. 3-25(b). ■

3-8 DON'T-CARE CONDITIONS

The logical sum of the minterms associated with a Boolean function specifies the conditions under which the function is equal to 1. The function is equal to 0 for the rest of the minterms. This assumes that all the combinations of the values for the variables of the function are valid. In practice, there are some applications where the function is not specified for certain combinations of the variables. As an example, the four-bit binary code for the decimal digits has six combinations that are not used and consequently are considered as unspecified. Functions that have unspecified outputs for some input combinations are called incompletely specified functions. In most applications, we simply don't care what value is assumed by the function for the unspecified minterms. For this reason, it is customary to call the unspecified minterms of a function don't-care condi-

tions. These don't-care conditions can be used on a map to provide further simplification of the Boolean expression.

It should be realized that a don't-care minterm is a combination of variables whose logical value is not specified. It cannot be marked with a 1 in the map because it would require that the function always be a 1 for such combination. Likewise, putting a 0 on the square requires the function to be 0. To distinguish the don't-care condition from 1's and 0's, an X is used. Thus, an X inside a square in the map indicates that we don't care whether the value of 0 or 1 is assigned to F for the particular minterm.

When choosing adjacent squares to simplify the function in a map, the don't-care minterms may be assumed to be either 0 or 1. When simplifying the function, we can choose to include each don't-care minterm with either the 1's or the 0's, depending on which combination gives the simplest expression.

Example 3-12

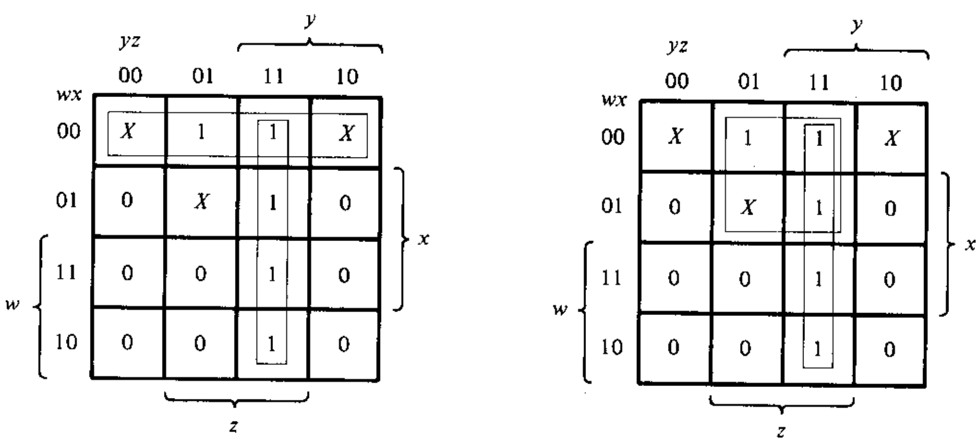
Simplify the Boolean function

$$F(w, x, y, z) = \Sigma(1, 3, 7, 11, 15)$$

that has the don't-care conditions

$$d(w, x, y, z) = \Sigma(0, 2, 5)$$

The minterms of F are the variable combinations that make the function equal to 1. The minterms of d are the don't-care minterms that may be assigned either 0 or 1. The map simplification is shown in Fig. 3-26. The minterms of F are marked by 1's, those of d are marked by X's, and the remaining squares are filled with 0's. To get the simplified expression in sum of products, we must include all the five 1's in the map, but



(a) $F = yz + w'x'$

(b) $F = yz + w'z$

FIGURE 3-26
Example with don't-care conditions

we may or may not include any of the X's, depending on the way the function is simplified. The term yz covers the four minterms in the third column. The remaining minterm m_1 can be combined with minterm m_3 to give the three-literal term $w'x'z$. However, by including one or two adjacent X's we can combine four adjacent squares to give a two-literal term. In part (a) of the diagram, don't-care minterms 0 and 2 are included with the 1's, which results in the simplified function

$$F = yz + w'x'$$

In part (b), don't-care minterm 5 is included with the 1's and the simplified function now is

$$F = yz + w'z$$

Either one of the above expressions satisfies the conditions stated for this example. ■

The above example has shown that the don't-care minterms in the map are initially marked with X's and are considered as being either 0 or 1. The choice between 0 and 1 is made depending on the way the incompletely specified function is simplified. Once the choice is made, the simplified function so obtained will consist of a sum of minterms that includes those minterms that were initially unspecified and have been chosen to be included with the 1's. Consider the two simplified expressions obtained in Example 3-12:

$$F(w, x, y, z) = yz + w'x' = \Sigma(0, 1, 2, 3, 7, 11, 15)$$

$$F(w, x, y, z) = yz + w'z = \Sigma(1, 3, 5, 7, 11, 15)$$

Both expressions include minterms 1, 3, 7, 11, and 15 that make the function F equal to 1. The don't-care minterms 0, 2, and 5 are treated differently in each expression. The first expression includes minterms 0 and 2 with the 1's and leaves minterm 5 with the 0's. The second expression includes minterm 5 with the 1's and leaves minterms 0 and 2 with the 0's. The two expressions represent two functions that are algebraically unequal. Both cover the specified minterms of the function, but each covers different don't-care minterms. As far as the incompletely specified function is concerned, either expression is acceptable since the only difference is in the value of F for the don't-care minterms.

It is also possible to obtain a simplified product of sums expression for the function of Fig. 3-26. In this case, the only way to combine the 0's is to include don't-care minterms 0 and 2 with the 0's to give a simplified complemented function:

$$F' = z' + wy'$$

Taking the complement of F' gives the simplified expression in product of sums:

$$F(w, x, y, z) = z(w' + y) = \Sigma(1, 3, 5, 7, 11, 15)$$

For this case, we include minterms 0 and 2 with the 0's and minterm 5 with the 1's.

3-9 THE TABULATION METHOD

The map method of simplification is convenient as long as the number of variables does not exceed five or six. As the number of variables increases, the excessive number of squares prevents a reasonable selection of adjacent squares. The obvious disadvantage of the map is that it is essentially a trial-and-error procedure that relies on the ability of the human user to recognize certain patterns. For functions of six or more variables, it is difficult to be sure that the best selection has been made.

The tabulation method overcomes this difficulty. It is a specific step-by-step procedure that is guaranteed to produce a simplified standard-form expression for a function. It can be applied to problems with many variables and has the advantage of being suitable for machine computation. However, it is quite tedious for human use and is prone to mistakes because of its routine, monotonous process. The tabulation method was first formulated by Quine and later improved by McCluskey. It is also known as the Quine–McCluskey method.

The tabular method of simplification consists of two parts. The first is to find by an exhaustive search all the terms that are candidates for inclusion in the simplified function. These terms are called *prime implicants*. The second operation is to choose among the prime implicants those that give an expression with the least number of literals.

3-10 DETERMINATION OF PRIME IMPLICANTS

The starting point of the tabulation method is the list of minterms that specify the function. The first tabular operation is to find the prime implicants by using a matching process. This process compares each minterm with every other minterm. If two minterms differ in only one variable, that variable is removed and a term with one less literal is found. This process is repeated for every minterm until the exhaustive search is completed. The matching-process cycle is repeated for those new terms just found. Third and further cycles are continued until a single pass through a cycle yields no further elimination of literals. The remaining terms and all the terms that did not match during the process comprise the prime implicants. This tabulation method is illustrated by the following example.

Example 3-13

Simplify the following Boolean function by using the tabulation method:

$$F = \Sigma(0, 1, 2, 8, 10, 11, 14, 15)$$

Step 1: Group binary representation of the minterms according to the number of 1's contained, as shown in Table 3-5, column (a). This is done by grouping the minterms into five sections separated by horizontal lines. The first section contains the number with no 1's in it. The second section contains those numbers that have only one 1. The

TABLE 3-5
Determination of Prime Implicants for Example 3-13

(a)			(b)			(c)	
	w x y z			w x y z			w x y z
0	0 0 0 0	√	0, 1	0 0 0 –		0, 2, 8, 10	– 0 – 0
			0, 2	0 0 – 0	√	0, 8, 2, 10	– 0 – 0
1	0 0 0 1	√	0, 8	– 0 0 0	√	10, 11, 14, 15	1 – 1 –
2	0 0 1 0	√				10, 14, 11, 15	1 – 1 –
8	1 0 0 0	√	2, 10	– 0 1 0	√		
			8, 10	1 0 – 0	√		
10	1 0 1 0	√					
			10, 11	1 0 1 –	√		
			10, 14	1 – 1 –	√		
11	1 0 1 1	√					
14	1 1 1 0	√					
			11, 15	1 – 1 1	√		
15	1 1 1 1	√	14, 15	1 1 1 –	√		

third, fourth, and fifth sections contain those binary numbers with two, three, and four 1's, respectively. The decimal equivalents of the minterms are also carried along for identification.

Step 2: Any two minterms that differ from each other by only one variable can be combined, and the unmatched variable removed. Two minterm numbers fit into this category if they both have the same bit value in all positions except one. The minterms in one section are compared with those of the next section down only, because two terms differing by more than one bit cannot match. The minterm in the first section is compared with each of the three minterms in the second section. If any two numbers are the same in every position but one, a check is placed to the right of both minterms to show that they have been used. The resulting term, together with the decimal equivalents, is listed in column (b) of the table. The variable eliminated during the matching is denoted by a dash in its original position. In this case, m_0 (0000) combines with m_1 (0001) to form (000–). This combination is equivalent to the algebraic operation

$$m_0 + m_1 = w'x'y'z' + w'x'y'z = w'x'y'$$

Minterm m_0 also combines with m_2 to form (00–0) and with m_8 to form (–000). The result of this comparison is entered into the first section of column (b). The minterms of sections two and three of column (a) are next compared to produce the terms listed in the second section of column (b). All other sections of (a) are similarly compared and subsequent sections formed in (b). This exhaustive comparing process results in the four sections of (b).

Step 3: The terms of column (b) have only three variables. A 1 under the variable means it is unprimed, a 0 means it is primed, and a dash means the variable is not included in the term. The searching and comparing process is repeated for the terms in

column (b) to form the two-variable terms of column (c). Again, terms in each section need to be compared only if they have dashes in the same position. Note that the term (000–) does not match with any other term. Therefore, it has no check mark at its right. The decimal equivalents are written on the left-hand side of each entry for identification purposes. The comparing process should be carried out again in column (c) and in subsequent columns as long as proper matching is encountered. In the present example, the operation stops at the third column.

Step 4: The unchecked terms in the table form the prime implicants. In this example, we have the term $w'x'y'$ (000–) in column (b), and the terms $x'z'$ (–0–0) and wy (1–1–) in column (c). Note that each term in column (c) appears twice in the table, and as long as the term forms a prime implicant, it is unnecessary to use the same term twice. The sum of the prime implicants gives a simplified expression for the function. This is because each checked term in the table has been taken into account by an entry of a simpler term in a subsequent column. Therefore, the unchecked entries (prime implicants) are the terms left to formulate the function. For the present example, the sum of prime implicants gives the minimized function in sum of products:

$$F = w'x'y' + x'z' + wy$$ ■

It is worth comparing this answer with that obtained by the map method. Figure 3-27 shows the map simplification of this function. The combinations of adjacent squares give the three prime implicants of the function. The sum of these three terms is the simplified expression in sum of products.

It is important to point out that Example 3-13 was purposely chosen to give the simplified function from the sum of prime implicants. In most other cases, the sum of prime implicants does not necessarily form the expression with the minimum number of terms. This is demonstrated in Example 3-14.

The tedious manipulation that one must undergo when using the tabulation method is reduced if the comparing is done with decimal numbers instead of binary. A method will now be shown that uses subtraction of decimal numbers instead of the comparing and matching of binary numbers. We note that each 1 in a binary number represents the

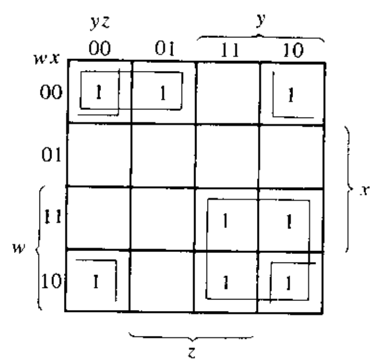


FIGURE 3-27

Map for the function of Example 3-13; $F = w'x'y' + x'z' + wy$

coefficient multiplied by a power of 2. When two minterms are the same in every position except one, the minterm with the extra 1 must be larger than the number of the other minterm by a power of 2. Therefore, two minterms can be combined if the number of the first minterm differs by a power of 2 from a second larger number in the next section down the table. We shall illustrate this procedure by repeating Example 3-13.

As shown in Table 3-6, column (a), the minterms are arranged in sections as before, except that now only the decimal equivalents of the minterms are listed. The process of comparing minterms is as follows: Inspect every two decimal numbers in adjacent sections of the table. If the number in the section below is *greater* than the number in the section above by a power of 2 (i.e., 1, 2, 4, 8, 16, etc.), check both numbers to show that they have been used, and write them down in column (b). The pair of numbers transferred to column (b) includes a third number in parentheses that designates the power of 2 by which the numbers differ. The number in parentheses tells us the position of the dash in the binary notation. The results of all comparisons of column (a) are shown in column (b).

The comparison between adjacent sections in column (b) is carried out in a similar fashion, except that only those terms with the same number in parentheses are compared. The pair of numbers in one section must differ by a power of 2 from the pair of numbers in the next section. And the numbers in the next section below must be *greater* for the combination to take place. In column (c), write all four decimal numbers with the two numbers in parentheses designating the positions of the dashes. A comparison of Tables 3-5 and 3-6 may be helpful in understanding the derivations in Table 3-6.

TABLE 3-6
Determination of Prime Implicants of Example 3-13 with Decimal Notation

(a)	(b)	(c)
0 ✓	0, 1 (1)	0, 2, 8, 10 (2, 8)
	0, 2 (2) ✓	0, 2, 8, 10 (2, 8)
1 ✓	0, 8 (8) ✓	
2 ✓		10, 11, 14, 15 (1, 4)
8 ✓	2, 10 (8) ✓	10, 11, 14, 15 (1, 4)
	8, 10 (2) ✓	
10 ✓		
	10, 11 (1) ✓	
11 ✓	10, 14 (4) ✓	
14 ✓		
	11, 15 (4) ✓	
15 ✓	14, 15 (1) ✓	

The prime implicants are those terms not checked in the table. These are the same as before, except that they are given in decimal notation. To convert from decimal notation to binary, convert all decimal numbers in the term to binary and then insert a dash in those positions designated by the numbers in parentheses. Thus 0, 1 (1) is converted to binary as 0000, 0001; a dash in the first position of either number results in (000–). Similarly, 0, 2, 8, 10 (2, 8) is converted to the binary notation from 0000, 0010, 1000, and 1010, and a dash inserted in positions 2 and 8, to result in (–0–0).

Example 3-14

Determine the prime implicants of the function

$$F(w, x, y, z) = \Sigma(1, 4, 6, 7, 8, 9, 10, 11, 15)$$

The minterm numbers are grouped in sections, as shown in Table 3-7, column (a). The binary equivalent of the minterm is included for the purpose of counting the number of

TABLE 3-7
Determination of Prime Implicants for Example 3-14

(a)			(b)			(c)
0001	1	√	1, 9	(8)		8, 9, 10, 11 (1, 2)
0100	4	√	4, 6	(2)		8, 9, 10, 11 (1, 2)
1000	8	√	8, 9	(1)	√	
			8, 10	(2)	√	
0110	6	√				
1001	9	√	6, 7	(1)		
1010	10	√	9, 11	(2)	√	
			10, 11	(1)	√	
0111	7	√				
1011	11	√	7, 15	(8)		
			11, 15	(4)		
1111	15	√				

Prime implicants

Decimal	Binary w	x	y	z	Term
1, 9 (8)	–	0	0	1	$x'y'z$
4, 6 (2)	0	1	–	0	$w'xz'$
6, 7 (1)	0	1	1	–	$w'xy$
7, 15 (8)	–	1	1	1	xyz
11, 15 (4)	1	–	1	1	wyz
8, 9, 10, 11 (1, 2)	1	0	–	–	wx'

1's. The binary numbers in the first section have only one 1; in the second section, two 1's; etc. The minterm numbers are compared by the decimal method and a match is found if the number in the section below is greater than that in the section above. If the number in the section below is smaller than the one above, a match is not recorded even if the two numbers differ by a power of 2. The exhaustive search in column (a) results in the terms of column (b), with all minterms in column (a) being checked. There are only two matches of terms in column (b). Each gives the same two-literal term recorded in column (c). The prime implicants consist of all the unchecked terms in the table. The conversion from the decimal to the binary notation is shown at the bottom of the table. The prime implicants are found to be $x'y'z$, $w'xz'$, $w'xy$, xyz, wyz, and wx'. ■

The sum of the prime implicants gives a valid algebraic expression for the function. However, this expression is not necessarily the one with the minimum number of terms. This can be demonstrated from inspection of the map for the function of Example 3-14. As shown in Fig. 3-28, the minimized function is recognized to be

$$F = x'y'z + w'xz' + xyz + wx'$$

which consists of the sum of four of the six prime implicants derived in Example 3-14. The tabular procedure for selecting the prime implicants that give the minimized function is the subject of the next section.

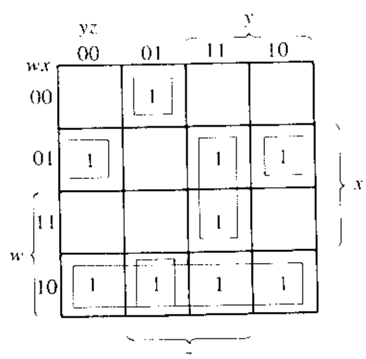


FIGURE 3-28
Map for the function of Example 3-14;
$F = x'y'z' + w'xz' + xyz + wx'$

3-11 SELECTION OF PRIME IMPLICANTS

The selection of prime implicants that form the minimized function is made from a prime implicant table. In this table, each prime implicant is represented in a row and each minterm in a column. X's are placed in each row to show the composition of

minterms that make the prime implicants. A minimum set of prime implicants is then chosen that covers all the minterms in the function. This procedure is illustrated in Example 3-15.

Example 3-15

Minimize the function of Example 3-14. The prime-implicant table for this example is shown in Table 3-8. There are six rows, one for each prime implicant (derived in Example 3-14), and nine columns, each representing one minterm of the function. X's are placed in each row to indicate the minterms contained in the prime implicant of that row. For example, the two X's in the first row indicate that minterms 1 and 9 are contained in the prime implicant $x'y'z$. It is advisable to include the decimal equivalent of the prime implicant in each row, as it conveniently gives the minterms contained in it. After all the X's have been marked, we proceed to select a minimum number of prime implicants.

The completed prime-implicant table is inspected for columns containing only a single X. In this example, there are four minterms whose columns have a single X: 1, 4, 8, and 10. Minterm 1 is covered by prime implicant $x'y'z$, i.e., the selection of prime implicant $x'y'z$ guarantees that minterm 1 is included in the function. Similarly, minterm 4 is covered by prime implicant $w'xz'$, and minterms 8 and 10, by prime implicant wx'. Prime implicants that cover minterms with a single X in their column are called *essential prime implicants*. To enable the final simplified expression to contain all the minterms, we have no alternative but to include essential prime implicants. A check mark is placed in the table next to the essential prime implicants to indicate that they have been selected.

Next we check each column whose minterm is covered by the selected essential prime implicants. For example, the selected prime implicant $x'y'z$ covers minterms 1 and 9. A check is inserted in the bottom of the columns. Similarly, prime implicant $w'xz'$ covers minterms 4 and 6, and wx' covers minterms 8, 9, 10, and 11. Inspection of the prime-implicant table shows that the selection of the essential prime implicants

TABLE 3-8
Prime Implicant Table for Example 3-15

		1	4	6	7	8	9	10	11	15
√ $x'y'z$	1, 9	X					X			
√ $w'xz'$	4, 6		X	X						
$w'xy$	6, 7			X	X					
xyz	7, 15				X					X
wyz	11, 15								X	X
√ wx'	8, 9, 10, 11					X	X	X	X	
		√	√	√		√	√	√	√	

covers all the minterms of the function except 7 and 15. These two minterms must be included by the selection of one or more prime implicants. In this example, it is clear that prime implicant xyz covers both minterms and is therefore the one to be selected. We have thus found the minimum set of prime implicants whose sum gives the required minimized function:

$$F = x'y'z + w'xz' + wx' + xyz$$ ■

The simplified expressions derived in the preceding examples were all in the sum of products form. The tabulation method can be adapted to give a simplified expression in product of sums. As in the map method, we have to start with the complement of the function by taking the 0's as the initial list of minterms. This list contains those minterms not included in the original function that are numerically equal to the maxterms of the function. The tabulation process is carried out with the 0's of the function and terminates with a simplified expression in sum of products of the complement of the function. By taking the complement again, we obtain the simplified product of sums expression.

A function with don't-care conditions can be simplified by the tabulation method after a slight modification. The don't-care terms are included in the list of minterms when the prime implicants are determined. This allows the derivation of prime implicants with the least number of literals. The don't-care terms are not included in the list of minterms when the prime implicant table is set up, because don't-care terms do not have to be covered by the selected prime implicants.

3-12 CONCLUDING REMARKS

Two methods of Boolean-function simplification were introduced in this chapter. The criterion for simplification was taken to be the minimization of the number of literals in sum of product or products of sums expressions. Both the map and the tabulation methods are restricted in their capabilities since they are useful for simplifying only Boolean functions expressed in the standard forms. Although this is a disadvantage of the methods, it is not very critical. Most applications prefer the standard forms over any other form. We have seen from Fig. 3-15 that the gate implementation of expressions in standard form consists of no more than two levels of gates. Expressions not in the standard form are implemented with more than two levels.

One should recognize that the Gray-code sequence chosen for the maps is not unique. It is possible to draw a map and assign a Gray-code sequence to the rows and columns different from the sequence employed here. As long as the binary sequence chosen produces a change in only one bit between adjacent squares, it will produce a valid and useful map.

Two alternate versions of the three-variable maps that are often found in the digital

logic literature are shown in Fig. 3-29. The minterm numbers are written in each square for reference. In (a), the assignment of the variables to the rows and columns is different from the one used in this book. In (b), the map has been rotated in a vertical position. The minterm number assignment in all maps remains in the order xyz. For example, the square for minterm 6 is found by assigning to the ordered variables the binary number $xyz = 110$. The square for this minterm is found in (a) from the column marked $xy = 11$ and the row with $z = 0$. The corresponding square in (b) belongs in the column marked with $x = 1$ and the row with $yz = 10$. The simplification procedure with these maps is exactly the same as described in this chapter except, of course, for the variations in minterm and variable assignment.

Two other versions of the four-variable map are shown in Fig. 3-30. The map in (a) is very popular and is used quite often in the literature. Here again, the difference is

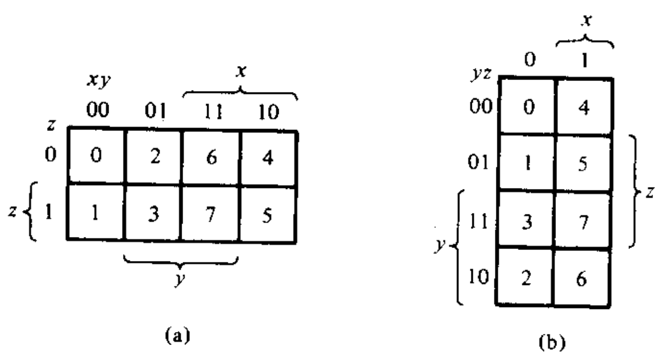


FIGURE 3-29
Variations of the three-variable map

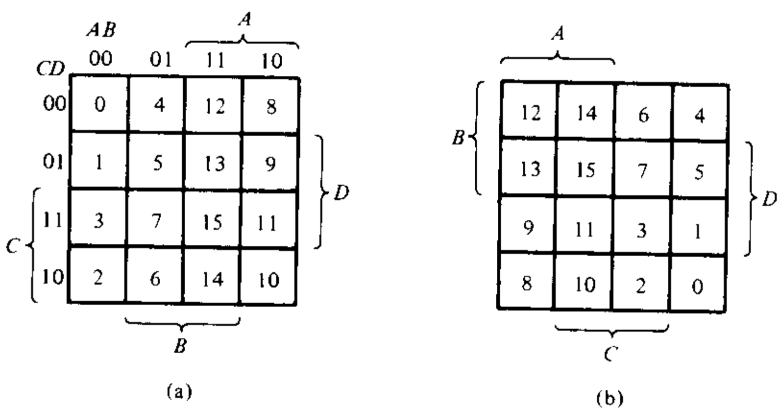


FIGURE 3-30
Variations of the four-variable map

slight and is manifested by a mere interchange of variable assignment from rows to columns and vice versa. The map in (b) is the original Veitch diagram that Karnaugh modified to the one shown in (a). Again, the simplification procedures do not change when these maps are used instead of the one employed in this book. There are also variations of the five-variable map. In any case, any map that looks different from the one used in this book, or is called by a different name, should be recognized merely as a variation of minterm assignment to the squares in the map.

As is evident from Examples 3-13 and 3-14, the tabulation method has the drawback that errors inevitably occur in trying to compare numbers over long lists. The map method would seem to be preferable, but for more than five variables, we cannot be certain that the best simplified expression has been found. The real advantage of the tabulation method lies in the fact that it consists of specific step-by-step procedures that guarantee an answer. Moreover, this formal procedure is suitable for computer mechanization.

In this chapter, we have considered the simplification of functions with many input variables and a single output variable. However, some digital circuits have more than one output. Such circuits are described by a set of Boolean functions, one for each output variable. A circuit with multiple outputs may sometimes have common terms among the various functions that can be utilized to form common gates during the implementation. This results in further simplification not taken into consideration when each function is simplified separately. There exists an extension of the tabulation method for multiple-output circuits. However, this method is too specialized and very tedious for human manipulation. It is of practical importance only if a computer program based on this method is available to the user.

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PROBLEMS

3-1 Simplify the following Boolean functions using three-variable maps:
(a) $F(x, y, z) = \Sigma(0, 1, 5, 7)$
(b) $F(x, y, z) = \Sigma(1, 2, 3, 6, 7)$
(c) $F(x, y, z) = \Sigma(3, 5, 6, 7)$
(d) $F(A, B, C) = \Sigma(0, 2, 3, 4, 6)$

3-2 Simplify the following Boolean expressions using three-variable maps:
(a) $xy + x'y'z' + x'yz'$
(b) $x'y' + yz + x'yz'$
(c) $A'B + BC' + B'C'$

3-3 Simplify the following Boolean functions using four-variable maps:
(a) $F(A, B, C, D) = \Sigma(4, 6, 7, 15)$
(b) $F(w, x, y, z) = \Sigma(2, 3, 12, 13, 14, 15)$
(c) $F(A, B, C, D) = \Sigma(3, 7, 11, 13, 14, 15)$

3-4 Simplify the following Boolean functions using four-variable maps:
(a) $F(w, x, y, z) = \Sigma(1, 4, 5, 6, 12, 14, 15)$
(b) $F(A, B, C, D) = \Sigma(0, 1, 2, 4, 5, 7, 11, 15)$
(c) $F(w, x, y, z) = \Sigma(2, 3, 10, 11, 12, 13, 14, 15)$
(d) $F(A, B, C, D) = \Sigma(0, 2, 4, 5, 6, 7, 8, 10, 13, 15)$

3-5 Simplify the following Boolean expressions using four-variable maps:
(a) $w'z + xz + x'y + wx'z$
(b) $B'D + A'BC' + AB'C + ABC'$
(c) $AB'C + B'C'D' + BCD + ACD' + A'B'C + A'BC'D$
(d) $wxy + yz + xy'z + x'y$

3-6 Find the minterms of the following Boolean expressions by first plotting each function in a map:
(a) $xy + yz + xy'z$
(b) $C'D + ABC' + ABD' + A'B'D$
(c) $wxy + x'z' + w'xz$

3-7 Simplify the following Boolean functions by first finding the essential prime implicants:
(a) $F(w, x, y, z) = \Sigma(0, 2, 4, 5, 6, 7, 8, 10, 13, 15)$
(b) $F(A, B, C, D) = \Sigma(0, 2, 3, 5, 7, 8, 10, 11, 14, 15)$
(c) $F(A, B, C, D) = \Sigma(1, 3, 4, 5, 10, 11, 12, 13, 14, 15)$

3-8 Simplify the following Boolean functions using five-variable maps:
(a) $F(A, B, C, D, E) = \Sigma(0, 1, 4, 5, 16, 17, 21, 25, 29)$
(b) $F(A, B, C, D, E) = \Sigma(0, 2, 3, 4, 5, 6, 7, 11, 15, 16, 18, 19, 23, 27, 31)$
(c) $F = A'B'CE' + A'B'C'D' + B'D'E' + B'CD' + CDE' + BDE'$

3-9 Simplify the following Boolean functions in product of sums:
(a) $F(w, x, y, z) = \Sigma(0, 2, 5, 6, 7, 8, 10)$
(b) $F(A, B, C, D) = \Pi(1, 3, 5, 7, 13, 15)$
(c) $F(x, y, z) = \Sigma(2, 3, 6, 7)$
(d) $F(A, B, C, D) = \Pi(0, 1, 2, 3, 4, 10, 11)$

3-10 Simplify the following expressions in (i) sum of products and (ii) products of sums:
(a) $x'z' + y'z' + yz' + xy$
(b) $AC' + B'D + A'CD + ABCD$
(c) $(A' + B' + D')(A + B' + C')(A' + B + D')(B + C' + D')$

3-11 Draw the AND-OR gate implementation of the following function after simplifying it in (a) sum of products and (b) product of sums:

$$F = (A, B, C, D) = \Sigma(0, 2, 5, 6, 7, 8, 10)$$

3-12 Simplify the following expressions and implement them with two-level NAND gate circuits:
(a) $AB' + ABD + ABD' + A'C'D' + A'BC'$
(b) $BD + BCD' + AB'C'D'$

3-13 Draw a NAND logic diagram that implements the complement of the following function:

$$F(A, B, C, D) = \Sigma(0, 1, 2, 3, 4, 8, 9, 12)$$

3-14 Draw a logic diagram using only two-input NAND gates to implement the following expression:

$$(AB + A'B')(CD' + C'D)$$

3-15 Simplify the following functions and implement them with two-level NOR gate circuits:
(a) $F = wx' + y'z' + w'yz'$
(b) $F(w, x, y, z) = \Sigma(5, 6, 9, 10)$

3-16 Implement the functions of Problem 3-15 with three-level NOR gate circuits [similar to Fig. 3-21(b)].

3-17 Implement the expressions of Problem 3-12 with three-level NAND circuits [similar to Fig. 3-19(c)].

3-18 Give three possible ways to express the function F with eight or fewer literals.

$$F(A, B, C, D) = \Sigma(0, 2, 5, 7, 10, 13)$$

3-19 Find eight different two-level gate circuits to implement

$$F = xy'z + x'yz + w$$

3-20 Implement the function F with the following two-level forms: NAND-AND, AND-NOR, OR-NAND, and NOR-OR.

$$F(A,B,C,D) = \Sigma(0, 1, 2, 3, 4, 8, 9, 12)$$

3-21 List the eight degenerate two-level forms and show that they reduce to a single operation. Explain how the degenerate two-level forms can be used to extend the number of inputs to a gate.

3-22 Simplify the following Boolean function F together with the don't-care conditions d; then express the simplified function in sum of minterms.
(a) $F(x, y, z) = \Sigma(0, 1, 2, 4, 5)$
$d(x, y, z) = \Sigma(3, 6, 7)$

(b) $F(A, B, C, D) = \Sigma(0, 6, 8, 13, 14)$
$d(A, B, C, D) = \Sigma(2, 4, 10)$
(c) $F(A, B, C, D) = \Sigma(1, 3, 5, 7, 9, 15)$
$d(A, B, C, D) = \Sigma(4, 6, 12, 13)$

3-23 Simplify the Boolean function F together with the don't-care conditions d in (i) sum of products and (ii) product of sums.
(a) $F(w, x, y, z) = \Sigma(0, 1, 2, 3, 7, 8, 10)$
$d(w, x, y, z) = \Sigma(5, 6, 11, 15)$
(b) $F(A, B, C, D) = \Sigma(3, 4, 13, 15)$
$d(A, B, C, D) = \Sigma(1, 2, 5, 6, 8, 10, 12, 14)$

3-24 A logic circuit implements the following Boolean function:

$$F = A'C + AC'D'$$

It is found that the circuit input combination $A = C = 1$ can never occur. Find a simpler expression for F using the proper don't-care conditions.

3-25 Implement the following Boolean function F together with the don't-care conditions d using no more than two NOR gates. Assume that both the normal and complement inputs are available.

$$F(A, B, C, D) = \Sigma(0, 1, 2, 9, 11)$$

$$d(A, B, C, D) = \Sigma(8, 10, 14, 15)$$

3-26 Simplify the following Boolean function using the map presented in Fig. 3-30(a). Repeat using the map of Fig. 3-30(b).

$$F(A, B, C, D) = \Sigma(1, 2, 3, 5, 7, 9, 10, 11, 13, 15)$$

3-27 Simplify the following Boolean functions by means of the tabulation method:
(a) $P(A, B, C, D, E, F, G) = \Sigma(20, 28, 52, 60)$
(b) $P(A, B, C, D, E, F, G) = \Sigma(20, 28, 38, 39, 52, 60, 102, 103, 127)$
(c) $P(A, B, C, D, E, F) = \Sigma(6, 9, 13, 18, 19, 25, 27, 29, 41, 45, 57, 61)$